Solution Key | January 2019
Financial Econometrics A

Question A:

Consider the time series model given by,

$$x_t = \sigma_t z_t, \quad t = 1, 2, ... \tag{A.1}$$

with $z_t \sim i.i.d.N(0, 1)$ and

$$\sigma_t^2 = \omega + \alpha x_{t-1}^2 + \beta g(y_{t-1}), \tag{A.2}$$

with $\omega > 0$, $\alpha, \beta \geq 0$. Here y_t is some exogenous covariate, as for example the realized volatility, and $g(\cdot)$ is a *continuous* function satisfying $g(y_t) \geq 0$ for all t. The initial x_0 and y_0 are taken as given.

Question A.1: Suppose that $\beta = 0$. State a condition for x_t to be weakly mixing and such that $Ex_t^2 < \infty$. You do not have to provide any derivations.

Solution: For $\beta = 0$, x_t is well-known ARCH(1) with Gaussian innovations. Clearly, x_t is a Markov chain with continuous transition density. In this case one can apply the drift criterion in order to show that x_t is weakly mixing with finite second-order moments if $\alpha < 1$. No derivations are needed.

Question A.2: Suppose that $\beta > 0$. Assume that y_t is $i.i.d.N\left(0, \sigma_y^2\right)$, and that the processes (z_t) and (y_t) are independent.
With $v_t = (x_t, y_t)'$ it holds that the density of v_t conditional on $(v_0, v_1, ..., v_{t-1})$ is given by

$$f(v_t|v_{t-1}, ..., v_0) = f(x_t|v_{t-1})f(y_t|v_{t-1}), \quad t \geq 1.$$

Argue that v_t is a Markov chain for which the transition density $f(\cdot|\cdot)$ is such that the drift criterion can be applied.

Next, suppose that $g(y_t) = y_t^2$. With $v = (x, y)'$ let $\|v\|^2 = v'v = x^2 + y^2$. With drift function $\delta(v_t) = 1 + \|v_t\|^2$, show that for some constant c,

$$E(\delta(v_t)|v_{t-1} = v) \leq c + \max(\alpha, \beta)\left(x^2 + y^2\right).$$

Conclude that if $\max(\alpha, \beta) < 1$, then v_t is weakly mixing with $E[x_t^2] + E[y_t^2] < \infty$.

Solution: Clearly, by the model structure, $f(v_t|v_{t-1}, ..., v_0) = f(v_t|v_{t-1}) = f(x_t|v_{t-1})f(y_t|v_{t-1})$, where

$$f(x_t|v_{t-1}) = \frac{1}{\sqrt{2\pi(\omega + \alpha x_{t-1}^2 + \beta g(y_{t-1}))}} \exp\left(-\frac{x_t^2}{2(\omega + \alpha x_{t-1}^2 + \beta g(y_{t-1}))}\right),$$

$$f(y_t|v_{t-1}) = f(y_t) = \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left(-\frac{y_t^2}{2\sigma_y^2}\right).$$

Hence $f\left(v_t|v_{t-1}\right)$ is strictly positive and continuous in v_t and v_{t-1}, since $g(\cdot) \geq 0$ is continuous

Next $E\left(\delta\left(v_t\right)|v_{t-1}=v\right) = 1+\omega+\alpha x^2+\beta y^2+\sigma_y^2 \leq 1+\omega+\sigma_y^2+\max\left(\alpha,\beta\right)\left(x^2+y^2\right)$, and the weakly mixing is shown by standard arguments for the drift criterion (e.g. letting $\|v\|^2 \to \infty$).

Question A.3: Suppose that y_t is not necessarily $i.i.d.N(0,\sigma_y^2)$ and that $g(y_t)$ is not necessarily equal to y_t^2.

Let $\theta = (\omega,\alpha,\beta)'$. With $L_T\left(\theta\right)$ the log-likelihood function for the model, it holds that the score for β is given by,

$$S\left(\theta\right) = \partial \log L_T\left(\theta\right)/\partial\beta = \sum_{t=1}^{T} \frac{1}{2}\left(\frac{x_t^2}{\sigma_t^2(\theta)} - 1\right)\frac{g(y_{t-1})}{\sigma_t^2(\theta)},$$

with

$$\sigma_t^2(\theta) = \omega + \alpha x_{t-1}^2 + \beta g(y_{t-1}).$$

Suppose that $(x_t, g(y_t))'$ is weakly mixing and that the true parameter values $\theta_0 = (\omega_0,\alpha_0,\beta_0)'$ satisfy $\omega_0 > 0$, $\alpha_0 < 1$, and $0 < \beta_L \leq \beta_0 < 1$. Show that as $T \to \infty$

$$\frac{1}{\sqrt{T}} S\left(\theta_0\right) \xrightarrow{d} N\left(0, \frac{\nu}{2}\right), \tag{A.3}$$

where

$$\nu = E\left[\left(\frac{g(y_{t-1})}{\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 g(y_{t-1})}\right)^2\right] \leq 1/\beta_L^2 < \infty.$$

Solution: (A.3) is shown by applying the CLT for weakly mixing processes. Note that

$$S\left(\theta_0\right) = \sum_{t=1}^{T} f_t,$$
$$f_t = \frac{1}{2}\left(\frac{x_t^2}{\sigma_t^2(\theta_0)} - 1\right)\frac{g(y_{t-1})}{\sigma_t^2(\theta_0)} = \frac{1}{2}\left(z_t^2 - 1\right)\frac{g(y_{t-1})}{\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 g(y_{t-1})}.$$

It holds that

$$E[f_t|x_{t-1}, g(y_{t-1})] = 0,$$

and

$$E[f_t^2] = \frac{1}{4}E[(z_t^2-1)^2]E\left[\left(\frac{g(y_{t-1})}{\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 g(y_{t-1})}\right)^2\right] = \frac{v}{2},$$

where

$$v = E\left[\left(\frac{g(y_{t-1})}{\omega_0 + \alpha_0 x_{t-1}^2 + \beta_0 g(y_{t-1})}\right)^2\right] \leq \beta_0^{-2} \leq \beta_L^{-2}.$$

Question A.4: Explain briefly what (A.3) can be used for.

Solution: (A.3) can be used for deriving the (limiting) distribution of the MLE. This can be used for addressing the estimation uncertainty of the model parameters, or hypothesis testing. Note that additional condtions are needed. Some details should be provided.

Question A.5: Suppose now that y_t is the square-root of the Realized Volatility based on 10-minutes intraday log-returns on the S&P500 Index, denoted $RV_t^{1/2}$. A plot of $RV_t^{1/2}$ and its sample autocorrelation function is given in Figure A.1.

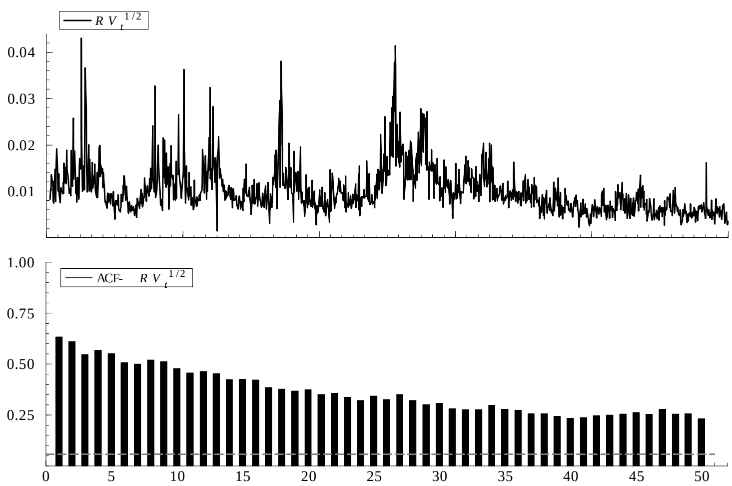

Figure A.1

Maximum likelihood estimation of the parameters of the model (A.1)-(A.2) with $g(RV_t^{1/2}) = RV_t^{1/2}$ gave the following output:

Output: MLE of ARCH with RV	
$\hat{\alpha} = 0.07$	std.deviation($\hat{\alpha}$) = 0.012
$\hat{\beta} = 0.21$	std.deviation($\hat{\beta}$) = 0.121

What would you conclude in terms of the importance of Realized Volatility?

Solution: Relevant to test the null hypothesis $H_0 : \beta = 0$. One obtains the t-statistic $t_{\beta=0} = \hat{\beta}/\text{std.deviation}(\hat{\beta}) = 0.21/0.121 \approx 1.75$. Based on conventional critical values (from the standard normal distribution), one rejects H_0 against the one-sided alternative $\beta > 0$ (which is the relevant alternative given the constraint $\beta \geq 0$). Note that no misspecification tests are given. In relation to the results in Question A.3-A.4, one may note that the true $\beta_0 > 0$, which rules out H_0. Moreover, the ACF in Figure A.1 indicates that $RV_t^{1/2}$ is highly persistent and hence may not be weakly mixing. Overall one should be cautious about drawing any conclusions based on the estimation output.

Question B:

Question B.1: As part of a discussion of "bubbles" in financial markets, consider the asset log-price series y_t in Figure B.1 with $t = 1, 2, ..., 1620$.

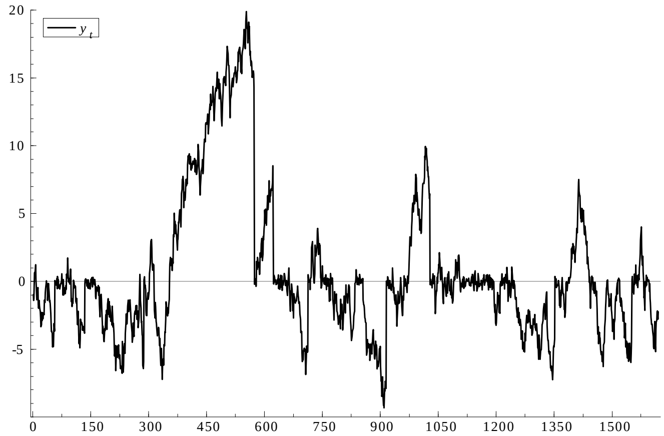

Figure B.1

For estimation, the following 2-state Markov switching model was applied:

$$y_t = \rho_{s_t} y_{t-1} + \sigma_{s_t} z_t, \quad t = 2, ..., T, \quad T = 1620. \tag{B.1}$$

Here z_t is $i.i.d.N(0,1)$ distributed and y_1 is fixed. Moreover, $s_t \in \{1, 2\}$ is an unobserved state variable governed by the transition matrix $P = (p_{ij})_{i,j=1,2}$ with $p_{ij} = P(s_t = j | s_{t-1} = i)$. The processes (z_t) and (s_t) are independent. It holds that

$$\rho_{s_t} = \rho 1 (s_t = 1) \quad \text{and} \quad \sigma^2_{s_t} = \sigma^2_1 1 (s_t = 1) + \sigma^2_2 1 (s_t = 2). \tag{B.2}$$

Gaussian likelihood estimation gave the following output, with misspecification tests in terms of smoothed standardized residuals $\hat{z}^*_t$:

MLE of P:	$\hat{p}_{11} = 0.98$ $\quad \hat{p}_{21} = 0.07$
MLE of ρ :	$\hat{\rho} = 0.99$
MLE of σ^2_1 and σ^2_2	$\hat{\sigma}^2_1 = 0.71$ and $\hat{\sigma}^2_2 = 0.30$
	p-values:
Test for Normality of $\hat{z}^*_t$:	0.12
LM-test for no ARCH in $\hat{z}^*_t$:	0.10
LR-test of $\rho = 1$:	0.81

Interpret the model. What would you conclude on the basis of the output and Figure B.1?

Solution: Two-state model. AR(1) with Gaussian errors in state 1, and Gaussian noise in state 2.
The misspecification tests based on $\hat{z}_t^*$ suggest that the model appears to be well-specified. Estimates of $\hat{p}_{11}$ and $\hat{p}_{22}$ suggest that each regime is quite persistent. Point estimate of autoregressive coefficient ρ, indicates a unit root process in regime 1. This is confirmed by the LR-test.

Question B.2: In order to obtain an estimate of $\theta = (p_{11}, p_{22}, \rho, \sigma_1^2, \sigma_2^2)$, the function $M(\theta)$ given by

$$M(\theta) = \sum_{i,j=1}^{2} \log p_{ij} \sum_{t=2}^{1620} p_t^*(i,j) + \sum_{j=1}^{2} \sum_{t=2}^{1620} p_t^*(j) \log f_\theta (y_t | y_{t-1}, s_t = j), \quad \text{(B.3)}$$

can be used. Provide an expression for $f_\theta (y_t | y_{t-1}, s_t = 1)$.

Solution: $f_\theta (y_t | y_{t-1}, s_t = 1) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \exp\left(-\frac{(y_t - \rho y_{t-1})^2}{2\sigma_1^2}\right)$. Details should be provided.

Question B.3: Explain how you would use $M(\theta)$ from (B.3) in order to find and estimate, $\hat{\theta}$, of θ.

Comment briefly on what Figure B.2 shows in relation to finding $\hat{\theta}$.

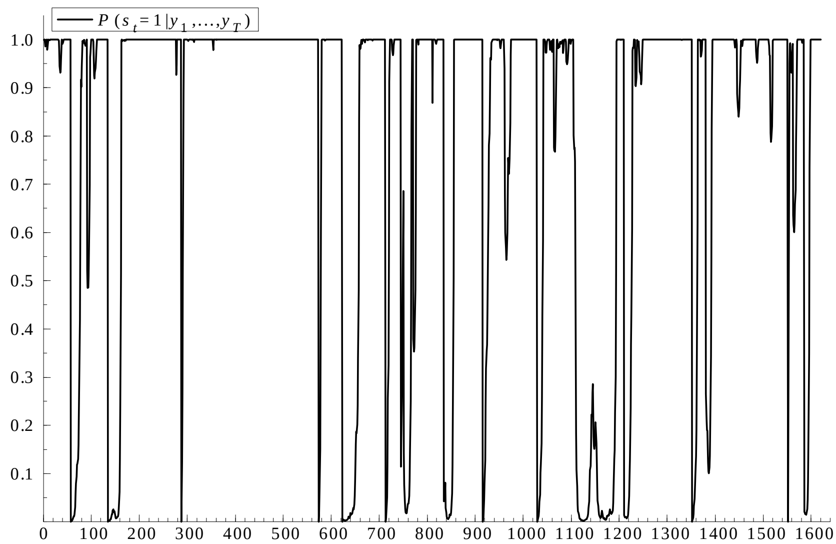

Figure B.2: $P(s_t = 1|y_1, ..., y_T)$

Solution: Use EM-algorithm to obtain estimates. This relies on computing the smoothed probabilites $p_t^*(i,j)$ and $p_t^*(j)$, where $p_t^*(1)$ is given in Figure B.2. Details should be provided.

Question B.4: Now assume that at time T, $s_T = 1$. In order to forecast if one will enter state 2 at $T+2$, derive

$$P(s_{T+2} = 2|s_T = 1),$$

and provide an estimate of this given the estimation output in Question B.1.

Solution: $P(s_{T+2} = 2|s_T = 1) = p_{22}p_{12} + p_{12}p_{11}$.
Based on the estimation output, we obtain the estimate $\hat{P}(s_{T+2} = 2|s_T = 1) = \hat{p}_{22}\hat{p}_{12} + \hat{p}_{12}\hat{p}_{11} = (1 - 0.07) \times (1 - 0.98) + (1 - 0.98) \times 0.98 = 3.82\%$.